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The Denjoy–Wolff Theorem for Condensing Holomorphic Mappings

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If B is the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $f: B \to B$ is holomorphic, condensing with respect to $\alpha_{\|\cdot\|}$, and fixed-point-free, then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of the iterates of f converges in the compact-open topology to the constant map taking the value ξ. © 1999 Academic Press

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1. INTRODUCTION

Let B be the open unit ball in a Banach space $(X, \|\cdot\|)$ and let $f: B \to B$ be a holomorphic, fixed-point-free self-map. In view of the Denjoy–Wolff theorem [12, 50] it is natural to ask if the sequence of iterates $\{f^n(x)\}$, where $x \in B$, converges. There is a large literature on the problem of iterating holomorphic, fixed-point-free mappings in $\mathbb{C}^n$. See, e.g., [1, 2, 26, 31, 39, 42], as well as [3, 7, 8, and 41] for interesting surveys and references. In the infinite dimensional case it is known [44] that the Denjoy–Wolff theorem fails even for biholomorphic self-maps of the unit ball. But if f is either a compact holomorphic self-map, a holomorphic automorphism of a special type, a firmly $k_{\mathbf{B}}$-nonexpansive mapping, or an averaged mapping of the first or second kind, then positive results were established in [11, 22, 23, 29, 30, 33, 40, 47, and 48] (see also [18, 49] for compact holomorphic self-maps of the open unit balls of J^*-algebras, and [16, 17] for analytic functions of operators). In this paper we use a new type of horosphere to

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establish the Denjoy–Wolff theorem for all fixed-point-free, $\alpha_{\|\cdot\|}$-condensing, and holomorphic self-maps of the open unit ball in any strictly convex Banach space.

2. PRELIMINARIES

All Banach spaces will be complex. If D is a bounded domain in a Banach space $(X, \|\cdot\|)$, then k_{D} always denotes its Kobayashi distance. We remark in passing that all distances assigned to a convex bounded domain D by Schwarz–Pick systems of pseudometrics [19, 23, 25] coincide [13, 38, 46]. A subset C of a bounded domain D is said to lie strictly inside D if $\operatorname{dist}(C, \partial D) > 0$. A mapping $f\colon D \to D$ is said to map D strictly inside D if $f(D)$ lies strictly inside D.

THEOREM 2.1 [25]. *Let D be a convex bounded domain in a Banach space $(X, \|\cdot\|)$. A subset C of D is k_{D}-bounded if and only if C is strictly inside D.*

The next theorem is due to Earle and Hamilton [14] (a proof for the Kobayashi distance can be found in [19]).

THEOREM 2.2 [14]. *Let D be a bounded domain in a Banach space $(X, \|\cdot\|)$. If a holomorphic $f\colon D \to D$ maps D strictly inside itself, then there exists $0 \leqslant t < 1$ such that*

$$k_{\mathrm{D}}(f(x), f(y)) \leqslant t k_{\mathrm{D}}(x, y)$$

for all x and y in D.

Let (Y, d) be a metric space and let $\varnothing \neq D \subset Y$. A mapping $f\colon D \to D$ is said to be d-nonexpansive if

$$d(f(x), f(y)) \leqslant d(x, y)$$

for all $x, y \in D$. Each holomorphic $f\colon D \to D$ is k_{D}-nonexpansive [19, 23, 25].

Hence, if D is a bounded convex domain in a Banach space $(X, \|\cdot\|)$, then by Theorem 2.2 the mapping $g_{s,z} = (1-s)z + s(\cdot)\colon D \to D$ is a k_{D}-contraction for every $z \in D$ and $0 \leqslant s < 1$. Thus for each k_{D}-nonexpansive mapping $f\colon D \to D$, the mapping $f_{s,z} = g_{s,z} \circ f = (1-s)z + sf\colon D \to D$ is a k_{D}-contraction and has exactly one fixed point, which we denote by $h_f(s, z)$. Fix $0 \leqslant s < 1$ and $x_0 \in D$. Then the mapping $h_f(s, \cdot)\colon D \to D$ is k_{D}-nonexpansive (holomorphic if f is holomorphic [10, 19, 27]) as a limit of the sequence $\{f^n_{s,\cdot}(x_0)\}$.

LEMMA 2.1 [29]. *Let B be the open unit ball in a Banach space $(X, \|\cdot\|)$ and let $k_{\mathbf{B}}$ denote its Kobayashi distance. For each $\xi \in \partial B$ and $0 < s < 1$ we have*

$$\lim_{y \to \xi} [k_{\mathbf{B}}(s\xi, y) - k_{\mathbf{B}}(0, y)] = -k_{\mathbf{B}}(s\xi, 0).$$

Observe that Lemma 2.1 is a generalization to all Banach spaces of the analogous result obtained by Yang [51] in the case of $\mathbb{C}^n$ (see also [1] and [29]).

If D is a bounded domain in a Banach space, then each holomorphic $f\colon D \to D$ is $k_{\mathbf{D}}$-nonexpansive. The intersection $\mathrm{Aut}(D) \cap C^0(\bar{D})$ will denote the group of all those biholomorphisms φ of D onto itself such that φ and φ^{-1} have 1–1 continuous extensions to the boundary ∂D.

Let (Y, d) be a metric space and let $\emptyset \neq D \subset Y$. We say that a mapping $f\colon D \to D$ is α_d-condensing with respect to Kuratowski's measure of noncompactness α_d [37] if

$$\alpha_d(f(A)) < \alpha_d(A)$$

for each bounded $A \subset D$ with $\alpha_d(A) > 0$. More information on condensing mappings and their applications can be found, for example, in [4–6, 21, 34].

We say that a metric space (X, ρ) is finitely totally bounded if each nonempty bounded subset of X is totally bounded. Finally, we recall Całka's theorem on the behavior of the sequence of iterates of a nonexpansive mapping on a finitely totally bounded metric space X.

THEOREM 2.3 [9]. *Let f be a nonexpansive mapping of a finitely totally bounded metric space X into itself. If for some $x_0 \in X$ the sequence $\{f^n(x_0)\}$ contains a bounded subsequence, then for every $x \in X$ the sequence $\{f^n(x)\}$ is bounded.*

3. HOROSPHERES

Let D be a convex bounded domain in a complex Banach space $(X, \|\cdot\|)$. In [1] Abate introduced the notion of horospheres in D (see also [51]), which was a generalization of the notion of ellipsoids in the open unit ball. Horospheres are useful tools in the investigation of the behavior of iterates of holomorphic mappings [8, 20, 22, 23, 28, 32, 48].

For $x \in D$, $\xi \in \partial D$, $R > 0$, $x_n \in D$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} x_n = \xi$, the small horosphere $E_x(\xi, R)$ and the big horosphere $F_x(\xi, R)$ of center x and radius R are defined by

$$E_x(\xi, R) = \{ y \in D : \limsup_{w \to \xi} [k_D(y, w) - k_D(x, w)] < \tfrac{1}{2} \log R\}$$

and

$$F_x(\xi, R) = \{ y \in D : \liminf_{w \to \xi} [k_D(y, w) - k_D(x, w)] < \tfrac{1}{2} \log R\}.$$

Another type of horosphere was defined in [29]. Let $x \in D$, $\xi \in \partial D$, $R > 0$, $x_n \in D$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} x_n = \xi$. Let us assume in addition that the limit

$$\lim_{n \to \infty} [k_D(y, x_n) - k_D(x, x_n)]$$

exists for all $y \in D$. Then the horosphere $G(x, \xi, R, \{x_n\})$ in D is defined as follows:

$$G(x, \xi, R, \{x_n\}) = \{ y \in D : \lim_{n \to \infty} [k_D(y, x_n) - k_D(x, x_n)] < \tfrac{1}{2} \log R\}.$$

Observe that if X is a separable Banach space, $x_n \in D$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} x_n = \xi$, then by a standard diagonalization procedure there exists a subsequence $\{x_{n_i}\}$ such that all the limits

$$\lim_{i \to \infty} [k_D(y, x_{n_i}) - k_D(x, x_{n_i})], \qquad x, y \in D,$$

exist. Therefore all the horospheres $G(x, \xi, R, \{x_{n_i}\})$ are well defined.

Now let $x \in D$, $\xi \in \partial D$, $R > 0$, $x_n \in D$, $n = 1, 2, \ldots$, and $\lim_{n \to \infty} x_n = \xi$. Fix a Banach limit $\mathrm{LIM} \in (l^\infty)^*$. The new horosphere $H(x, \xi, R, \{x_n\})$ in D is defined as follows:

$$H(x, \xi, R, \{x_n\}) = \{ y \in D : \mathrm{LIM}[k_D(y, x_n) - k_D(x, x_n)] < \tfrac{1}{2} \log R\}.$$

Since

$$|k_D(y, x_n) - k_D(x, x_n)| \leqslant k_D(y, x)$$

for each $n = 1, 2, \ldots$, the horosphere $H(x, \xi, R, \{x_n\})$ is well defined.

THEOREM 3.1. *Let D be a convex bounded domain in a Banach space $(X, \|\cdot\|)$. Let $x \in D$, $\xi \in \partial D$, $R > 0$, $x_n \in D$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} x_n = \xi$. Then the horospheres $H(x, \xi, R, \{x_n\})$ have the following properties*:

(i) *$E_x(\xi, R) \subset H(x, \xi, R, \{x_n\}) \subset F_x(\xi, R)$ for every $x \in D$, $\xi \in \partial D$, and $R > 0$*;

(ii) *if the horosphere $G(x, \xi, R, \{x_n\})$ exists, then $G(x, \xi, R, \{x_n\}) = H(x, \xi, R, \{x_n\})$*;

(iii) *if $H(x, \xi, R, \{x_n\})$ is nonempty, then it is convex*;

(iv) *if $\varphi, \varphi^{-1} \in \mathrm{Aut}(D) \cap C^0(\bar{D})$, then*

$$\varphi(H(x, \xi, R, \{x_n\})) = H(\varphi(x), \varphi(\xi), R, \{\varphi(x_n)\});$$

(v) *if $\tilde{x}, \tilde{\tilde{x}} \in D$ and $\mathrm{LIM}[k_{\mathrm{D}}(\tilde{\tilde{x}}, x_n) - k_{\mathrm{D}}(\tilde{x}, x_n)] = \frac{1}{2}\log L$, then*

$$H(\tilde{\tilde{x}}, \xi, R, \{x_n\}) \subset H(\tilde{x}, \xi, LR, \{x_n\});$$

(vi) *for each $0 < R_1 < R_2$ we have*

$$[D \cap \overline{H(x, \xi, R_1, \{x_n\})}^{\|\cdot\|}] \subset H(x, \xi, R_2, \{x_n\});$$

(vii) *for every $R > 1$ we have $B(x, \frac{1}{2}\log R) \subset H(x, \xi, R, \{x_n\})$*;

(viii) *for every $R < 1$ we have $B(x, -\frac{1}{2}\log R) \cap H(x, \xi, R, \{x_n\}) = \varnothing$*;

(ix) *$\bigcup_{R>0} H(x, \xi, R, \{x_n\}) = D$ and $\bigcap_{R>0} H(x, \xi, R, \{x_n\}) = \varnothing$*;

(x) *if $D = B$, where B is the open unit ball in X, then*

$$H(x, \xi, R, \{x_n\}) \neq \varnothing$$

for each $R > 0$ and

$$\xi \in \bar{B}^{\|\cdot\|} \cap \bigcap_{R>0} \overline{H(x, \xi, R, \{x_n\})}^{\|\cdot\|} \subset \partial B;$$

(xi) *if $D = B$, where B is the open unit ball in a strictly convex Banach space X, then*

$$\partial B \cap \bigcap_{R>0} \overline{H(x, \xi, R, \{x_n\})}^{\|\cdot\|} = \bar{B}^{\|\cdot\|} \cap \bigcap_{R>0} \overline{H(x, \xi, R, \{x_n\})}^{\|\cdot\|} = \{\xi\}.$$

(xii) *If X is a Hilbert space and D is the open unit ball B in X, then*

$$E_0(\xi, R) = G(0, \xi, R, \{x_n\}) = H(0, \xi, R, \{x_n\}) = F_0(\xi, R).$$

Proof. Assertions (i)–(iv) and (vi)–(ix) are obvious.

(v) By the equality

$$\begin{aligned}
&\mathrm{LIM}[k_{\mathbf{D}}(y, x_n) - k_{\mathbf{D}}(\tilde{x}, x_n)] \\
&\quad = \mathrm{LIM}[k_{\mathbf{D}}(y, x_n) - k_{\mathbf{D}}(\tilde{\tilde{x}}, x_n)] + \mathrm{LIM}[k_{\mathbf{D}}(\tilde{\tilde{x}}, x_n) - k_{\mathbf{D}}(\tilde{x}, x_n)] \\
&\quad = \mathrm{LIM}[k_{\mathbf{D}}(y, x_n) - k_{\mathbf{D}}(\tilde{\tilde{x}}, x_n)] + \tfrac{1}{2}\log L,
\end{aligned}$$

we get

$$H(\tilde{\tilde{x}}, \xi, R, \{x_n\}) \subset H(\tilde{x}, \xi, LR, \{x_n\}),$$

as we claimed.

(x) It is sufficient to observe that by Lemma 2.1,

$$\begin{aligned}
&\mathrm{LIM}[k_{\mathbf{B}}(s\xi, x_n) - k_{\mathbf{B}}(x, x_n)] \\
&\quad = \mathrm{LIM}[k_{\mathbf{B}}(s\xi, x_n) - k_{\mathbf{B}}(0, x_n)] + \mathrm{LIM}[k_{\mathbf{B}}(0, x_n) - k_{\mathbf{B}}(x, x_n)] \\
&\quad = \lim_{n\to\infty}[k_{\mathbf{B}}(s\xi, x_n) - k_{\mathbf{B}}(0, x_n)] + \lim_{n\to\infty}[k_{\mathbf{B}}(0, x_n) - k_{\mathbf{B}}(x, x_n)] \\
&\quad = -k_{\mathbf{B}}(0, s\xi) + k_{\mathbf{B}}(0, x)
\end{aligned}$$

for each $0 < s < 1$.

(xi) This assertion follows directly from (x).

(xii) When X is a Hilbert space the following equality is valid [29] (see also [1] and [51] for $X = \mathbb{C}^n$):

$$\lim_{w\to\xi}[k_{\mathbf{B}}(x, w) - k_{\mathbf{B}}(0, w)] = \frac{1}{2}\log\frac{|1 - \langle x, \xi\rangle|}{1 - \|x\|^2}.$$

This yields the claimed result. ∎

Open Problem I. Let D be a convex bounded domain in a complex Banach space $(X, \|\cdot\|)$. Is the horosphere $H(x, \xi, R, \{x_n\})$ nonempty for every $R > 0$?

4. CONDENSING MAPPINGS

In this section we collect basic properties of condensing and $k_{\mathbf{D}}$-nonexpansive mappings which we need in the proof of our Denjoy–Wolff theorem. We begin with two properties of the Kobayashi distance.

LEMMA 4.1. *Let D be a convex bounded domain in a Banach space $(X, \|\cdot\|)$.*

(a) *If $x, y, w, z \in D$ and $s \in [0, 1]$, then*

$$k_D(sx+(1-s)\,y, sw+(1-s)\,z) \leqslant \max[k_D(x, w), k_D(y, z)];$$

(b) *if $x, y \in D$ and $s, t \in [0, 1]$, then*

$$k_D(sx+(1-s)\,y, tx+(1-t)\,y) \leqslant k_D(x, y).$$

Proof. Part (a) follows from the basic properties of the Kobayashi distance [36, 45].

(b) It is sufficient to observe that for $0 \leqslant s' \leqslant 1$ and $w, z \in D$ we have

$$k_D(s'w+(1-s')\,z, z) = k_D(s'w+(1-s')\,z, s'z+(1-s')\,z) \leqslant k_D(w, z)$$

by (a) and

$$sx+(1-s)\,y = \frac{s-t}{1-t}\,x + \left(1-\frac{s-t}{1-t}\right)[tx+(1-t)\,y]$$

for $0 \leqslant s \leqslant 1$; $0 \leqslant t < 1$; $t \leqslant s$; and $x, y \in D$. ∎

We say that a bounded convex domain D in a Banach space is strictly convex if for every $x, y \in \bar{D}^{\|\cdot\|}$ the open segment

$$(x, y) = \{z \in X : z = sx+(1-s)\,y \text{ for some } 0<s<1\}$$

lies in D.

LEMMA 4.2. *Let D be a strictly convex bounded domain in a Banach space $(X, \|\cdot\|)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in D which converge to $\xi \in \partial D$ and to $\eta \in \bar{D}$, respectively. If*

$$\sup\{k_D(x_n, y_n) : n = 1, 2, ...\} = c < \infty,$$

then $\xi = \eta$.

Proof. Assume $\xi \neq \eta$. Then by the strict convexity of D, each point of the open segment

$$(\xi, \eta) = \{z \in X : z = s\xi+(1-s)\,\eta \text{ for some } 0<s<1\}$$

lies in D, and for $0<s<1$,

$$\lim_{n\to\infty} k_D(s\xi+(1-s)\,\eta, sx_n+(1-s)\,y_n) = 0.$$

Hence, applying Lemma 4.1 we get

$$\begin{aligned} k_{\mathrm{D}}(s\xi+(1-s)\,\eta,\, t\xi+(1-t)\,\eta) &= \lim_{n\to\infty} k_{\mathrm{D}}(sx_n+(1-s)\,y_n,\, tx_n+(1-t)\,y_n) \\ &\leqslant \limsup_{n\to\infty} k_{\mathrm{D}}(x_n, y_n) \leqslant c < \infty \end{aligned}$$

for all $s, t \in (0, 1)$. By Theorem 2.1 we obtain that (ξ, η) lies strictly inside D, but this is impossible since $\xi \in \partial D$. ∎

LEMMA 4.3. *Let D be a nonempty bounded subset of a metric space (Y, d). If $f: D \to D$ is condensing with respect to α_d and $\{x_n\}$ is a sequence of elements of D such that $d(x_n, f(x_n)) \to 0$, then the set of all elements of the sequence $\{x_n\}$ is totally bounded in (Y, d).*

Proof. Since f is condensing with respect to α_d and

$$\alpha_d(\{x_n : n=1, 2, ...\}) = \alpha_d(\{f(x_n) : n=1, 2, ...\}),$$

the set $\{x_n : n=1, 2, ...\}$ must be totally bounded in (Y, d). ∎

COROLLARY 4.1. *Let D be a convex bounded domain in a Banach space $(X, \|\cdot\|)$. If $f: D \to D$ is k_{D}-nonexpansive and condensing with respect to $\alpha_{\|\cdot\|}$; C is a nonempty, k_{D}-closed, and f-invariant subset of D; $\{s_n\}$ is a sequence such that $\lim_{n\to\infty} s_n = 1$, $0 < s_n < 1$; and $\{z_n\}$ is a sequence of elements of C, then the sequence $\{x_n\}$ given by $x_n = h_f(s_n, z_n) = f_{s_n, z_n}(x_n) = (1-s_n)\, z_n + s_n f(x_n)$ for each n contains a norm-convergent subsequence.*

Proof. It is obvious that $f(x_n) - x_n \to 0$. Thus it is sufficient to apply Lemma 4.3. ∎

LEMMA 4.4. *Let (D, ρ) and (Y, d) be two metric spaces such that*

(i) *$D \subset Y$;*

(ii) *D is d-bounded;*

(iii) *both (D, ρ) and (Y, d) are complete;*

(iv) *for every $x \in D$ and every sequence $\{x_n\}$ in D which tends to $y \in Y \setminus D$ in (Y, d), we have $\lim_{n\to\infty} \rho(x, x_n) = \infty$;*

(v) *the topologies on D given by ρ and $d_{|D\times D}$ coincide.*

Let $f: D \to D$ be a ρ-nonexpansive mapping which is also condensing with respect to α_d. Then

(a) *for each* x *in* D, *each subsequence* $\{f^{n_i}(x)\}$ *of the sequence of iterates* $\{f^n(x)\}$ *contains a d-convergent subsequence*;

(b) *the mapping* f *has* ρ*-bounded orbits if and only if there exists an* $x \in D$ *with a* ρ*-bounded subsequence of iterates* $\{f^{n_i}(x)\}$;

(c) *if the mapping* f *has a* ρ*-unbounded orbit, then, for each* $x \in D$, *each subsequence of iterates* $\{f^{n_i}(x)\}$ *contains a further subsequence* $\{f^{n_{i_j}}(x)\}$ *which d-converges to* $y \in Y \backslash D$ *as* $j \to \infty$.

Proof. (a) In [43] Sadovskii proved that if $f\colon D \to D$ is a condensing (with respect to α_d) mapping, then the set $\{f^n(x) : n = 1, 2, \ldots\}$ is totally bounded.

(b) Assume that the subsequence $\{f^{n_i}(x)\}$ is ρ-bounded. By (a) we can apply Theorem 2.3 to conclude that all the orbits of f on D are ρ-bounded.

(c) This part follows directly from (a) and (b). ∎

THEOREM 4.1. *Let* D *be a convex bounded domain in a Banach space* $(X, \|\cdot\|)$ *and let* k_D *denote its Kobayashi distance. If* $f\colon D \to D$ *is* k_D*-nonexpansive and condensing with respect to* $\alpha_{\|\cdot\|}$, *then the following conditions are equivalent*:

(i) *f has a fixed point*;

(ii) *there exist* $x \in D$ *and a* k_D*-bounded subsequence of its iterates* $\{f^{n_i}(x)\}$;

(iii) *there exists* $x \in D$ *with a* k_D*-bounded sequence of its iterates* $\{f^n(x)\}$;

(iv) *for each* $x \in D$ *the sequence of its iterates* $\{f^n(x)\}$ *is* k_D*-bounded*;

(v) *there exists a nonempty,* k_D*-closed, convex,* k_D*-bounded, and f-invariant subset* C *of* D;

(vi) *there exists a nonempty,* k_D*-bounded, and f-invariant subset* C *of* D;

(vii) *there exists a* k_D*-bounded and norm-convergent sequence* $\{x_n\}$ *such that* $(f(x_n) - x_n) \to 0$;

(viii) *there exists a* k_D*-bounded sequence* $\{x_n\}$ *such that* $(f(x_n) - x_n) \to 0$.

Proof. The implication (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii) This follows from Lemma 4.4 (b).

(iii) ⇒ (iv) It is sufficient to apply the nonexpansiveness of f.

(iv) $\Rightarrow$ (v) Assume that $\{f^n(x_0)\}$ is k_{D}-bounded. We apply the method of asymptotic centers [15, 21, 23]. For every $y \in C$ and every k_{D}-bounded sequence the number

$$r(y, \{x_n\}) = \limsup_{n \to \infty} k_{\mathrm{D}}(y, x_n)$$

is called the asymptotic radius of $\{x_n\}$ at y and the number

$$r(\{x_n\}) = \inf_{y \in D} r(y, \{x_n\})$$

is the asymptotic radius of $\{x_n\}$ in D. Now we simply set

$$C = \{y \in D : r(y, \{f^n(x_0)\}) \leqslant r(\{f^n(x_0)\}) + 1\}$$

to get a nonempty, k_{D}-closed, convex, k_{D}-bounded and f-invariant subset C of D.

The implications (v) $\Rightarrow$ (vi) and (vi) $\Rightarrow$ (iii) are obvious.

(v) $\Rightarrow$ (vii) By Corollary 4.1, the nonempty, k_{D}-closed, convex, k_{D}-bounded, and f-invariant subset C of D contains a norm-convergent and k_{D}-bounded sequence $\{x_n\}$ such that $(f(x_n) - x_n) \to 0$.

(vii) $\Rightarrow$ (viii) This is obvious.

(vii) $\Rightarrow$ (i) If $\{x_n\}$ is a norm-convergent and k_{D}-bounded sequence such that

$$(f(x_n) - x_n) \to 0,$$

then its norm-limit belongs to D and is a fixed point of f.

(viii) $\Rightarrow$ (v) If $\{x_n\}$ is a k_{D}-bounded sequence such that

$$(f(x_n) - x_n) \to 0,$$

then

$$C = \{y \in D : r(y, \{x_n\}) \leqslant r(\{x_n\}) + 1\}$$

is a nonempty, k_{D}-closed, convex, k_{D}-bounded and f-invariant subset C of D. ∎

Remark 4.1. The assumption that $f: D \to D$ is condensing with respect to $\alpha_{\|\cdot\|}$ is essential as examples in [35] show.

5. THE DENJOY–WOLFF THEOREM

THEOREM 5.1. *If B is the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $f\colon B \to B$ is holomorphic, condensing with respect to $\alpha_{\|\cdot\|}$, and fixed-point-free, then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges in the compact-open topology to the constant map taking the value ξ.*

Proof. The mapping f, as a holomorphic one, is $k_{\mathbf{D}}$-nonexpansive. Since f is fixed-point-free, Corollary 4.1 and Theorem 4.1 show that there is a sequence

$$\{x_n\} = \{h_f(s_n, z_n)\} = \{f_{s_n, z_n}(x_n)\} = \{(1-s_n)\, z_n + s_n\, f(x_n)\}$$

($z_n \in B$ and $0 < s_n < 1$ for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} s_n = 1$) which is convergent to $\xi \in \partial B$. Let us observe that for arbitrary $y \in B$,

$$\begin{aligned}
&\mathrm{LIM}[k_{\mathbf{B}}(f(y), x_n) - k_{\mathbf{B}}(0, x_n)] \\
&\quad = \mathrm{LIM}[k_{\mathbf{B}}(f(y), f_{t_n, z_n}(y)) + k_{\mathbf{B}}(f_{t_n, z_n}(y), f_{t_n, z_n}(x_n)) - k_{\mathbf{B}}(0, x_n)] \\
&\quad \leqslant \mathrm{LIM}[k_{\mathbf{B}}(y, x_n) - k_{\mathbf{B}}(0, x_n)].
\end{aligned}$$

The above inequality shows that

$$f(H(0, \xi, R, \{x_n\})) \subset H(0, \xi, R, \{x_n\})$$

for arbitrary $R > 0$. Once more, since f is fixed-point-free, Theorem 4.1 implies that for each $x \in B$ we have

$$\lim_{n\to\infty} \|f^n(x)\| = 1.$$

Let $A \subset \partial B$ denote the set of all accumulation points of the sequence $\{f^n(x)\}$. By Lemma 4.4(c), A is nonempty. By Lemma 4.2, the set A is independent of the choice of $x \in B$ and therefore, by Theorem 3.1(xi),

$$\varnothing \neq A \subset \partial B \cap \bigcap_{R>0} \overline{H(0, \xi, R, \{x_n\})}^{\|\cdot\|} = \{\xi\}.$$

Since by Lemma 4.4(c), from every subsequence $\{f^{n_i}(x)\}$ of the sequence $\{f^n(x)\}$ we can choose a further norm convergent subsequence $\{f^{n_{i_j}}(x)\}$, the equality $A = \{\xi\}$ implies that $\lim_{n\to\infty} f^n(x) = \xi$. By Lemmas 4.2 and 4.4(c), the sequence $\{f^n\}$ is convergent in the compact-open topology to the constant map ξ. ∎

Remark 5.1. The assumption that $(X, \|\cdot\|)$ is a strictly convex Banach space is essential as examples in [1] and [11] show.

Remark 5.2. Note that our result is new even in an infinite-dimensional Hilbert space.

In the case of uniformly convex Banach spaces we are able to establish locally uniform convergence [19] of the sequence of iterates. To this end, let us recall the following lemma.

LEMMA 5.1 [29]. *Let X be a uniformly convex Banach space with the open unit ball B. If $w_n, z_n \in B$ for $n = 1, 2, \ldots$, $\lim_n z_n = \xi \in \partial B$, and*

$$\sup_n k_{\mathbf{B}}(z_n, w_n) < \infty,$$

then $\|z_n - w_n\| \to 0$.

Now we are ready to prove the following theorem.

THEOREM 5.2. *Let X be a uniformly convex Banach space with the open unit ball B. Let a homomorphic fixed-point-free $f\colon B \to B$ be condensing with respect to $\alpha_{\|\cdot\|}$. Then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ converges locally uniformly on B to the constant map taking the value ξ.*

Proof. It is sufficient to observe that if $\lim_n f^n(0) = \xi \in \partial B$, then for each sequence $\{x_i\}$ with $\|x_i\| \leqslant r < 1$, for $i = 1, 2, \ldots$, and every strictly increasing sequence of natural numbers $\{n_i\}$ we have

$$\sup_i k_{\mathbf{B}}(f^{n_i}(0), f^{n_i}(x_i)) \leqslant k_{\mathbf{B}}(0, x_i) \leqslant \operatorname{arg\,tanh} r < \infty. \quad \blacksquare$$

Since the proofs of Theorems 5.1 and 5.2 have a strictly metric character we can extend them to $k_{\mathbf{B}}$-nonexpansive mappings.

THEOREM 5.3. *If B is the open unit ball of a strictly convex Banach space $(X, \|\cdot\|)$ and $f\colon B \to B$ is $k_{\mathbf{B}}$-nonexpansive, condensing with respect to $\alpha_{\|\cdot\|}$, and fixed-point-free, then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ of iterates of f converges in the compact–open topology to the constant map taking the value ξ.*

THEOREM 5.4. *Let X be a uniformly convex Banach space with the open unit ball B. Let a $k_{\mathbf{B}}$-nonexpansive fixed-point-free $f\colon B \to B$ be condensing with respect to $\alpha_{\|\cdot\|}$. Then there exists $\xi \in \partial B$ such that the sequence $\{f^n\}$ converges locally uniformly on B to the constant map taking the value ξ.*

Remark 5.3. In Theorems 5.1, 5.2, 5.3, and 5.4 the Kuratowski measure of noncompactness can be replaced by the Hausdorff measure of noncompactness [4–6, 21].

Open Problem II. Are Theorems 5.1 and 5.3 valid for an arbitrary bounded strictly convex domain in a Banach space?

Let us observe that if Open Problem I has a positive answer, then we can repeat step by step the proof of Theorem 5.1 to get a positive answer for Open Problem II.

We finish our paper by observing that Corollary 4.1 and the method used in the proof of Theorem 5.1 yield the following proposition.

PROPOSITION 5.1. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space with the open unit ball B. If $f\colon B \to B$ is $k_{\mathbf{B}}$-nonexpansive, condensing with respect to $\alpha_{\|\cdot\|}$, and fixed-point-free, then there exists $\xi \in \partial B$ such that $\{h_f(s, \cdot)\}$ converges uniformly on B as $s \to 1$ to the constant map taking the value ξ. This ξ is the unique limit point of the sequence of iterates $\{f^n\}$.*

Remark 5.4. In Hilbert balls it is known that the approximating curves $\{h_f(\cdot, x)\}$ for fixed-point-free $k_{\mathbf{B}}$-nonexpansive mappings behave much better than the sequence of iterates $\{f^n(x)\}$. Namely, they always converge to the so-called "sink point" on the boundary even if the sequences of iterates do not (see [20, 23, 24, and 43]).

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